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$D = 10$, $N = IIB$ supergravity: Lorentz–invariant actions and duality

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Abstract: We present a manifestly Lorentz invariant and supersymmetric component field action for $D = 10$, type IIB supergravity, using a newly developed method for the construction of actions with chiral bosons, which implies only a single scalar non propagating auxiliary field. With the same method we construct also an action in which the complex two–form gauge potential and its Hodge–dual, a complex six–form gauge potential, appear in a symmetric way in compatibility with supersymmetry and Lorentz invariance. The duals of the two physical scalars of the theory turn out to be described by a $SL(2,\mathbb{R})$ triplet of eight–forms whose curvatures are constrained by a single linear relation. We present also a supersymmetric action in which the basic fields and their duals, six–form and eight–form potentials, appear in a symmetric way. All these actions are manifestly invariant under the global $SL(2,\mathbb{R})$–duality group of $D = 10$, IIB supergravity and are equivalent to each other in that their dynamics corresponds to the well known equations of motion of $D = 10$, IIB supergravity.

Keywords: String Duality, Space-Time Symmetries, Supergravity Models.

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Contents

1. Introduction and Summary

The obstacle which prevented for a long time a Lagrangian formulation of $D = 10$, $N = IIB$ supergravity is the appearance of a chiral boson in the spectrum of the theory, i.e. a four–form gauge potential with a self–dual field strength. A possible way to overcome this obstacle was presented in [1], by extending Siegel's action for two–dimensional chiral bosons [2] to higher dimensions. In this approach one gets, through lagrangian multipliers, not the self–duality condition as equation of motion for the chiral bosons, but rather its square. However, in dimensions greater than two, the elimination of the lagrangian multipliers seems problematic [2] and, moreover, at the quantum level, for example in the derivation of the Lorentz anomaly, it seems to present untreatable technical and conceptual problems. On the other hand, a group manifold action for $D = 10, IIB$ supergravity has been obtained in [3]. In general, when a group manifold action is restricted to ordinary space–time one gets a consistent supersymmetric action for the component fields; but if a self–dual (or anti self–dual) tensor is present, the restricted action loses supersymmetry [4].

A new method for writing manifestly Lorentz–invariant and supersymmetric actions for chiral bosons (p–forms) in $D = 2$ mod 4 has been presented in [5]: it uses a single non propagating auxiliary scalar field and involves two new bosonic symmetries; one of them allows to eliminate the auxiliary field and the other kills half of the degrees of freedom of the p–form, reducing it to a chiral boson. This method turned out to be compatible with all relevant symmetries, including supersymmetry and κ–symmetry [6, 7] and admits also a canonical coupling to gravity, being manifestly Lorentz invariant. As all lagrangian formulations of theories with chiral bosons, the method is expected to be insufficient for what concerns the quantization of these actions on manifolds with non–trivial topology [8], but it can be successfully applied, even at the quantum level, on trivial manifolds. As an example of the efficiency of this method also at the quantum level, we mention that the effective action for chiral bosons in two dimensions, coupled to a background metric, can easily be computed in a covariant way [9], and that it gives the expected result, namely the effective action of a two–dimensional complex Weyl–fermion. This implies, in turn, that also the Lorentz– and Weyl–anomalies due to a $D=2$ chiral boson, as derived with this new method, coincide with the ones predicted by the index theorem. Work regarding the Lorentz–anomaly in higher dimensions is in progress.

In this paper we present a manifestly Lorentz–invariant and supersymmetric action for $D=10$, $N=IIB$ supergravity, based on this method.[1] Apart from the above mentioned new features, the basic ingredients are the equations of motion and SUSY–transformations of the basic fields, which are well known [10, 11, 12], and can be most conveniently derived in a superspace approach [10, 3]. In addition to the metric, the bosonic fields in this theory are two complex scalars (0–forms), which parametrize the coset $\dfrac{SL(2,\mathbb{R})}{U(1)}$, a complex two–form gauge potential and the real chiral four–form gauge potential.

Since the discovery of D–branes and their coupling to RR gauge potentials, the Hodge duals to the zero– and two–forms (the four–form is self–dual) i.e. the eight– and six–forms acquired a deeper physical meaning. It is therefore of some interest to look at a Lagrangian formulation with manifest duality i.e. in which the zero and eight–forms and the two and six–forms appear in a symmetric way. The method presented in [5] appears particularly suitable to cope also with this problem. Indeed, a variant [14, 15] of this method allowed in the past to construct a manifestly duality invariant Lagrangian for Maxwell's equations in four dimensions [14] as well as for $N=1, D=11$ supergravity [15].

In this paper we shall also construct an action with manifest duality between the gauge potentials and their Hodge duals. Upon gauge fixing the new bosonic symmetries, mentioned above, one can remove the six– and eight–forms and recover the (standard) formulation with only zero– and two–forms. On the other hand, a Lagrangian formulation in which only the six and eight–forms appear, instead of the zero and two–forms, is not accessible for intrinsic reasons i.e. the presence of Chern–Simons forms in the definition of the curvatures.

The case of the eight–forms requires a second variant of the method. As we will see, manifest invariance under the global S–duality group $SL(2,\mathbb{R})$ of the theory requires the introduction of *three* real eight–form potentials, with three nine–form curvatures,

[1]The covariant action for the bosonic sector of type IIB supergravity has already been presented in [13].

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which belong to the adjoint representation of $SL(2,\mathbb{R})$. Two $SL(2,\mathbb{R})$–invariant combinations of the three nine–form curvatures are related by Hodge–duality to the two real (or one complex) one–form curvatures of the scalars. The third one is determined by an $SL(2,\mathbb{R})$–invariant linear constraint between the three curvature nine–forms. While the first variant of our method allows to treat *Hodge–duality relations* between forms at a Lagrangian level, the second variant allows to deal, still at a Lagrangian level, with *linear relations* between curvatures. This is then precisely what is needed to describe the dynamics of the eight–forms through an action principle.

The general validity of the method is underlined also by the fact that all these lagrangians are supersymmetric. To achieve supersymmetry one has to modify the SUSY transformation laws of the fermions in a very simple and canonical way, the modifications being proportional to the equations of motion, derived e.g. in a superspace approach. The on–shell SUSY algebra can then be seen to close on the two new bosonic symmetries mentioned at the beginning.

In section 2 we present the superspace language and results for $D = 10$, $N = IIB$ supergravity, following mainly [10]. There, we construct also the dual supercurvatures and potentials in a $SL(2,\mathbb{R})$ covariant way. These results are used at the *component level*, in section 4, to write a manifestly Lorentz–invariant action for the theory, using only the scalars, two and four–forms. In section 3 we give a concise account of the new method itself (for more details see [5, 6]). In section 5 we write an action in which the two and six–form potentials appear in a symmetric way and prove its invariance under supersymmetry. Section 6 is devoted to the construction of an action in which all gauge potentials appear in a symmetric way, paying special attention to the new features exhibited by the eight–forms. Section 7 collects some concluding remarks and observations.

2. Superspace results

The superspace conventions and results of this paper follow mainly [10]. The $D = 10$, IIB superspace is parametrized by the supercoordinates $Z^M = (x^m, \theta^\mu, \bar{\theta}^\mu)$ where the θ^μ are sixteen complex anticommuting coordinates. Here and in what follows the "bar" indicates simply complex conjugation and in case transposition. The cotangent superspace basis is indicated by $e^A = dZ^M e_M{}^A(Z) = (e^a, e^\alpha, e^{\bar{\alpha}}) \equiv (e^a, \psi^\alpha, \overline{\psi^\alpha})$, where $a = 0, 1, \ldots, 9$ and $\alpha = 1, \ldots, 16$, and $\psi^\alpha = dZ^M \psi_M{}^\alpha$ indicates the complex gravitino one–superform. All superforms can be decomposed along this basis. The Lorentz superconnection one–form is given by $\omega_a{}^b = dZ^M \omega_{M\,a}{}^b$ with curvature $R_a{}^b = d\omega_a{}^b + \omega_a{}^c \omega_c{}^b$.

The two physical real scalars of the theory parametrize the coset $\dfrac{SU(1,1)}{U(1)}$, where $SU(1,1) \approx SL(2,\mathbb{R})$ is the global S–duality symmetry group of $D = 10$, IIB supergravity, and the $U(1)$ is realized locally. The coset is described by two complex scalars

$(U, -\overline{V}) \equiv A_0$ which are constrained by $|U|^2 - |V|^2 = 1$ such that the matrix

$$W \equiv \begin{pmatrix} U & V \\ \overline{V} & \overline{U} \end{pmatrix} \tag{2.1}$$

belongs to $SU(1,1)$ and the fields $A_0 \equiv (U, -\overline{V})$ form an $SU(1,1)$ doublet. The Maurer–Cartan form $W^{-1}dW$ decomposes then as

$$W^{-1}dW = \begin{pmatrix} 2iQ & R_1 \\ \bar{R}_1 & -2iQ \end{pmatrix}, \tag{2.2}$$

where R_1 and Q are $SU(1,1)$ invariant one–forms:

$$R_1 = \overline{U}dV - Vd\overline{U}, \tag{2.3}$$

$$Q = \frac{1}{2i}(\overline{U}dU - Vd\overline{V}). \tag{2.4}$$

Since the $U(1)$ weights of (U, V) are $(-2, 2)$, i.e. A_0 has weight -2, Q is a $U(1)$–connection and R_1, which has to be considered as the curvature of the scalars, has $U(1)$ weight 4. We can then introduce a $U(1)$ and $SO(1,9)$ covariant derivative which acts on a p–form with $U(1)$ weight q as

$$D = d + \omega + qiQ. \tag{2.5}$$

For a list of the $U(1)$ weights and $SU(1,1)$ representations of the fields see the table at the end of the section.

For a p–form with purely bosonic components

$$\phi_p = \frac{1}{p!}e^{a_1}\dots e^{a_p}\phi_{a_p\dots a_1},$$

we introduce its Hodge–dual, a $(10-p)$–form, as

$$*(\phi_p) \equiv \frac{1}{(10-p)!}e^{a_1}\dots e^{a_{10-p}}(*\phi)_{a_{10-p}\dots a_1}, \tag{2.6}$$

where

$$(*\phi)_{a_1\dots a_{10-p}} \equiv \frac{1}{p!}\epsilon_{a_1\dots a_{10-p}}{}^{b_1\dots b_p}\phi_{b_1\dots b_p}.$$

In particular, on a p –form we have

$$(*)^2 = (-1)^{p+1}.$$

The other bosonic degrees of freedom are carried by the following superforms. We introduce a complex two–form A_2, where $(A_2, \bar{A}_2)$ constitute an $SU(1,1)$ doublet, and its dual which is a complex six–form A_6, where $(iA_6, \overline{iA_6})$ constitutes also a $SU(1,1)$ doublet. The real four–form in the theory, the "chiral boson", is denoted by A_4. As anticipated in the introduction the duals of the scalars are parametrized by three real eight–forms which are described by a complex eight–form A_8 and a purely imaginary one $\hat{A}_8$. The three forms $\left(A_8, \bar{A}_8, \hat{A}_8\right)$ form an $SU(1,1)$ triplet, i.e. they belong to the adjoint representation of $SU(1,1)$. All these forms are $U(1)$ singlets.

The curvatures associated to these forms maintain their $SU(1,1)$ and $U(1)$ representations and are given by

$$S_3 = dA_2\,, \qquad dS_3 = 0\,, \tag{2.7}$$

$$S_5 = dA_4 + i(A_2 d\bar{A}_2 - \bar{A}_2 dA_2)\,, \qquad dS_5 = 2i\bar{S}_3 S_3\,, \tag{2.8}$$

$$S_7 = dA_6 - \frac{i}{3}(S_5 + 2dA_4)A_2\,, \qquad dS_7 = iS_3 S_5\,, \tag{2.9}$$

$$S_9 = dA_8 + \left[\bar{S}_7 - \frac{i}{4}\bar{A}_2(S_5 + dA_4)\right]\bar{A}_2\,, \qquad dS_9 = \bar{S}_7\bar{S}_3\,, \tag{2.10}$$

$$\hat{S}_9 = d\hat{A}_8 + \frac{1}{2}\left[S_7 + \frac{i}{4}A_2(S_5 + dA_4)\right]\bar{A}_2 - c.c.\,, \qquad d\hat{S}_9 = \frac{1}{2}(S_7\bar{S}_3 - \bar{S}_7 S_3)\,. \tag{2.11}$$

Again $\left(S_9, \bar{S}_9, \hat{S}_9\right)$ form an $SU(1,1)$ triplet.

The superspace parametrizations of these curvatures are more conveniently given in terms of the $SU(1,1)$ invariant combinations which one can form using the scalars U and V. Including also the curvatures for the scalars these invariant curvatures are given by

$$R_1 = \overline{U}dV - Vd\overline{U}\,, \tag{2.12}$$

$$R_3 = \overline{U}S_3 + V\bar{S}_3\,, \tag{2.13}$$

$$R_5 = S_5, \tag{2.14}$$

$$R_7 = \overline{U}S_7 - V\bar{S}_7\,, \tag{2.15}$$

$$R_9 = \overline{U}^2\bar{S}_9 - V^2 S_9 + 2\overline{U}V\hat{S}_9\,, \tag{2.16}$$

$$\hat{R}_9 = UVS_9 - \overline{UV}\bar{S}_9 - \left(|U|^2 + |V|^2\right)\hat{S}_9\,. \tag{2.17}$$

R_1 and R_9 carry $U(1)$ charge 4, R_3 and R_7 carry charge 2 and R_5 and $\hat{R}_9$ carry charge 0 and are respectively real and purely imaginary, while all other R_n are complex. The associated Bianchi identities are

$$DR_1 = 0 \tag{2.18}$$

$$DR_3 = \bar{R}_3 R_1\,, \tag{2.19}$$

$$DR_5 = 2i\bar{R}_3 R_3\,, \tag{2.20}$$

$$DR_7 = -\bar{R}_7 R_1 + iR_3 R_5\,, \tag{2.21}$$

$$DR_9 = 2R_1\hat{R}_9 + R_7 R_3\,, \tag{2.22}$$

$$D\hat{R}_9 = \bar{R}_1 R_9 - R_1\bar{R}_9 + \frac{1}{2}\left(\bar{R}_7 R_3 - R_7\bar{R}_3\right). \tag{2.23}$$

For the $U(1)$ connection we have

$$dQ = \frac{i}{2}\bar{R}_1 R_1\,; \tag{2.24}$$

it is also useful to notice that

$$
\begin{aligned}
DU &= V\bar{R}_1\,, \\
DV &= UR_1\,.
\end{aligned}
\tag{2.25}
$$

Defining the torsion as usual by

$$
T^A = De^A, \tag{2.26}
$$

it satisfies the Bianchi identities

$$
DT^\alpha = \psi^\gamma R_\gamma{}^\alpha + \frac{1}{2}\bar{R}_1 R_1 \psi^\alpha\,, \tag{2.27}
$$

$$
DT^a = e^b R_b{}^a\,. \tag{2.28}
$$

The superspace parametrizations of the curvatures in (2.12)–(2.17) can now be written, for $n = 1, 3, 5, 7, 9, \hat{9}$, as

$$
R_n = F_n - C_n\,, \tag{2.29}
$$

where F_n indicates the purely bosonic part

$$
F_n = \frac{1}{n!} e^{a_1} \ldots e^{a_n} F_{a_n \ldots a_1}\,, \tag{2.30}
$$

and the n–forms C_n involve the gravitino one–form ψ^α and the complex spinor Λ_α, which completes the fermionic degrees of freedom of $D = 10$, IIB supergravity (contraction of spinorial indices is understood):

$$
C_1 = 2\psi\Lambda\,, \tag{2.31}
$$

$$
C_3 = \frac{1}{2} e^a e^b \left(\bar{\psi}\Gamma_{ab}\Lambda\right) + \frac{i}{2} e^a (\psi\Gamma_a\psi)\,, \tag{2.32}
$$

$$
C_5 = -\frac{1}{3!} e^a e^b e^c \left(\bar{\psi}\Gamma_{abc}\psi\right) + \frac{1}{5!} e^{a_1} \ldots e^{a_5} \left(\bar{\Lambda}\Gamma_{a_1 \ldots a_5}\Lambda\right), \tag{2.33}
$$

$$
C_7 = \frac{1}{6!} e^{a_1} \ldots e^{a_6} \left(\bar{\psi}\Gamma_{a_1 \ldots a_6}\Lambda\right) - \frac{i}{2}\frac{1}{5!} e^{a_1} \ldots e^{a_5} \left(\psi\Gamma_{a_1 \ldots a_5}\psi\right), \tag{2.34}
$$

$$
C_9 = \frac{2}{8!} e^{a_1} \ldots e^{a_8} (\psi\Gamma_{a_1 \ldots a_8}\Lambda)\,, \tag{2.35}
$$

$$
\hat{C}_9 = -\frac{i}{2}\left[\frac{e^{a_1} \ldots e^{a_7}}{7!}\left(\bar{\psi}\Gamma_{a_1 \ldots a_7}\psi\right) + \frac{6}{9!} e^{a_1} \ldots e^{a_9} \left(\bar{\Lambda}\Gamma_{a_1 \ldots a_9}\Lambda\right)\right]. \tag{2.36}
$$

Actually, C_5 and $\hat{C}_9$ contain also a contribution with only bosonic vielbeins. These amount, however, only to a redefinition of F_5 and $\hat{F}_9$. These redefinitions are convenient for what follows, see eqs. (2.42)–(2.45) below. It is also convenient to decompose the forms C_n as

$$
C_n = C_n^\Lambda + C_n^\psi\,, \tag{2.37}
$$

where C_n^Λ indicates the parts which depend on Λ_α and C_n^ψ the parts which are independent of Λ_α, in particular $C_1^\psi = 0 = C_9^\psi$.

The parametrizations of the torsions and of $D\Lambda_\alpha$ become

$$\begin{aligned}
De^a &= i\bar{\psi}\Gamma^a\psi\,, &(2.38)\\
D\psi &= \frac{1}{2}e^a e^b T_{ba} - (\bar{\psi}\Lambda)\bar{\psi} - \frac{1}{2}(\bar{\psi}\Gamma^a\bar{\psi})\Gamma^a\Lambda + \\
&\quad + ie^a\left[-\frac{21}{2}X_a + \frac{3}{2}X^b\Gamma_{ab} + \frac{5}{4}X_{abc}\Gamma^{bc} - \frac{1}{4}X^{bcd}\Gamma_{abcd} + \right.\\
&\quad \left. + \frac{1}{192}(F^+_{abcde} - 4X_{abcde})\Gamma^{bcde}\right]\psi + \frac{3}{16}e^a\left(-F_{abc}\Gamma^{bc} + \frac{1}{9}F^{bcd}\Gamma_{abcd}\right)\bar{\psi}\,, &(2.39)\\
D\Lambda &= e^b D_b\Lambda + \frac{i}{2}F^a\Gamma_a\bar{\psi} + \frac{i}{24}F^{abc}\Gamma_{abc}\psi\,. &(2.40)
\end{aligned}$$

Here $T_{ab}{}^\alpha$ parametrizes the part of $D\psi^\alpha$ with only bosonic vielbeins and

$$X^{(n)} \equiv \frac{1}{16}\bar{\Lambda}\Gamma^{(n)}\Lambda\,. \tag{2.41}$$

$F^+_{a_1\ldots a_5}$ indicates the self–dual part of $F_{a_1\ldots a_5}$, i.e. $F^\pm_{a_1\ldots a_5} = \frac{1}{2}(F_{a_1\ldots a_5} \pm (*F)_{a_1\ldots a_5})$. For the parametrization of $R_a{}^b$, see [10].

All these parametrizations and the form of the Bianchi identities are dictated by the consistency of the Bianchi identities for the torsion and for the curvatures R_n themselves.

Since the closure of the SUSY–algebra sets the theory on shell one gets also the following (self)–duality relations between the curvatures, which become extremely simple when expressed through the F_n defined in (2.29)-(2.36):

$$\begin{aligned}
*\,F_5 &= F_5\,, &(2.42)\\
*\,F_7 &= F_3\,, &(2.43)\\
*\,F_9 &= F_1\,, &(2.44)\\
*\,\hat{F}_9 &= 0\,. &(2.45)
\end{aligned}$$

The first relation is the equation of motion for the four–form A_4. Equation (2.43), which relates the curvature of A_6 to the curvature of A_2, promotes the Bianchi identities (2.19) and (2.21) to equations of motion for A_6 and A_2 ,respectively.

Equation (2.45) constitutes the linear constraint between $d\hat{A}_8$ and dA_8 mentioned in the introduction, and allows to express — through (2.17) — the curvature $\hat{S}_9$, and hence $\hat{A}_8$, as a function of S_9. Substituting this expression for $\hat{S}_9$ in (2.16) one can compute R_9 and F_9 as a function of dA_8. At this point the duality relation (2.44) promotes (2.18) and (2.22) to equations of motion for A_8 and $A_0 = (U, -\overline{V})$ respectively. The complex eight–form A_8 is thus dual to the two real scalars contained in U and V. These fields are,in fact, constrained by $|U|^2 - |V|^2 = 1$ and are subjected to the local $U(1)$ invariance. Once this invariance is fixed only two real physical scalars survive.

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It is clear that the elimination of $\hat{A}_8$ breaks manifest $SU(1,1)$ invariance and that a manifestly $SU(1,1)$ invariant action principle for the dual scalars has to be based on three eight–forms, i.e. (2.10)–(2.11), (2.16)–(2.17) and (2.44)–(2.45).

The occurrence of three eight–forms can also be understood from the following point of view. Since the theory possesses a global $SU(1,1)$ invariance there must exist three conserved currents which belong to the adjoint representation of $SU(1,1)$. The Hodge duals of these currents, which have to be closed and hence locally exact, are just given by dA_8 and $d\hat{A}_8$ and their explicit expressions can be derived from (2.10) and (2.11) using (2.44) and (2.45).

An expression for the dual curvatures and their Bianchi identities was given also in [16], in a non–manifestly $SU(1,1)$ and $U(1)$ covariant formulation. In this formulation it is sufficient to introduce only *two* eight–form potentials because the $U(1)$ invariance has been gauge fixed.

The equations of motion for the gravitino, for Λ_α and for the metric can be found in [10]; their explicit expressions are not needed here since the action is completely determined by SUSY invariance, by the knowledge of the Bianchi identities and by the superspace parametrizations given above.

The $SU(1,1)$ and $U(1)$ representations of the basic fields and their charges are:

	e^a	ψ^α	Λ_α	A_0	$(A_{2,6},\bar{A}_{2,6})$	$\left(A_8,\bar{A}_8,\hat{A}_8\right)$	$R_5,\hat{R}_9$	$R_3,\ R_7$	$R_1,\ R_9$
$n[U(1)]$	0	1	3	-2	0	0	0	2	4
$SU(1,1)$	1	1	1	2	2	3	1	1	1

3. The covariant method

From now on we will work in *ordinary* space–time but still continue to use the language of forms to avoid the explicit appearance of Lorentz indices. In particular, our actions will be written as integrals over ten–forms. In the next section we will perform the reduction of the superspace results of the preceding section to ordinary bosonic space–time.

In this section we will present the basic ingredients which allow to write covariant actions for equations (2.42)-(2.45) concentrating in particular on the self–duality equation of motion (2.42).

This equation is of the type

$$F_5^- \equiv \frac{1}{2}(F_5 - *F_5) = 0\,, \tag{3.1}$$

where

$$F_5 = dA_4 + \tilde{C}_5 \tag{3.2}$$

and $\tilde{C}_5$ is independent of A_4.[2]

[2]In the case of $D=10$, IIB supergravity we have $\tilde{C}_5 = C_5 + i(A_2 d\bar{A}_2 - \bar{A}_2 dA_2)$, now in ordinary space–time.

The covariant method [5] requires the introduction of a scalar auxiliary field $a(x)$ and the related vector

$$v_a(x) = \frac{e_a{}^m \partial_m a}{\sqrt{-g^{mn}\partial_m a \partial_n a}} \equiv e_a{}^m v_m\,, \tag{3.3}$$

satisfying $v^a v_a = -1$. We introduce also the one–form

$$v = e^a v_a = \frac{da}{\sqrt{-g^{mn}\partial_m a \partial_n a}}\,, \tag{3.4}$$

and indicate with i_v the interior product of a p–form with the vector field $v^m\partial_m$.

Defining

$$f_4 \equiv i_v(F_5 - *F_5)\,, \tag{3.5}$$

the action which reproduces (3.1) can be written as

$$\begin{aligned} S_0[A_4, a] &= \frac{1}{2}\int\left[\frac{1}{2}(F_5 * F_5 + f_4 * f_4) + \tilde{C}_5 dA_4\right] \\ &= \frac{1}{2}\int\left[\frac{1}{2}(F_5 * F_5 + f_4 * f_4) + F_5 dA_4\right]. \end{aligned} \tag{3.6}$$

The form of this action is selected, and fixed, by the following symmetries:

$$I)\quad \delta A_4 = \Lambda_3\, da, \qquad \delta a = 0\,, \tag{3.7}$$

$$II)\quad \delta A_4 = -\frac{\phi}{\sqrt{-(\partial a)^2}} f_4 \quad \delta a = \phi\,, \tag{3.8}$$

where ϕ and Λ_3 are transformation parameters, respectively a scalar and a three–form. The action S_0 is, actually, invariant also under *finite* transformations of the type I) i.e. under $A_4 \to A_4 + \Lambda_3\, da$. This fact becomes relevant in what follows.

The equation of motion for a and A_4 are respectively given by

$$d\left(\frac{1}{\sqrt{-(\partial a)^2}} v f_4 f_4\right) = 0, \tag{3.9}$$

$$d(v f_4) = 0\,, \tag{3.10}$$

The symmetry II) promotes the auxiliary field a to a "pure gauge" field and allows to gauge–fix[3] it to an arbitrary function $a(x) = a_0(x)$ provided that $g^{mn}\partial_m a_0\, \partial_n a_0 \neq 0$. Correspondingly, the equation of motion (3.9) can easily be seen to be a consequence of (3.10).

The general solution of (3.10) is $v f_4 = d\tilde{\Lambda}_3\, da$. Since under a finite transformation I) we have

$$v f_4 \to v f_4 + d\Lambda_3\, da,$$

[3]Typical non–covariant gauges are $a_0(x) = n_m x^m$ where n_m is a constant vector.

choosing $\Lambda_3 = \tilde{\Lambda}_3$ we get

$$f_4 = 0\,. \tag{3.11}$$

Due to the identity decomposition on a p–form

$$\mathbb{1} = (-1)^p v i_v - * v i_v * \,, \tag{3.12}$$

one gets the identity

$$F_5 - *F_5 = -v f_4 + *(v f_4) \tag{3.13}$$

and hence (3.11) is equivalent to the self–duality equation of motion for A_4 (3.1).

This concludes the proof that the action S_0 describes indeed interacting ($\tilde{C}_5 \neq 0$) chiral bosons in ten dimensions. If the fields composing $\tilde{C}_5$ are themselves dynamical, one has to complete the action S_0 by adding terms which involve the kinetic and interaction terms for those fields, but not A_4 itself because otherwise the symmetries I) and II) are destroyed.

Another five–form, which will acquire an important role in establishing supersymmetry invariance, is given by

$$K_5 \equiv F_5 + v f_4\,. \tag{3.14}$$

This five–form is uniquely determined by the following properties: it is self–dual,

$$K_5 = *K_5,$$

as follows from (3.13), and it reduces to F_5^+ if the self–duality constraint for A_4 (3.1) holds. K_5 constitutes therefore a kind of off–shell generalization of F_5^+.

4. The complete action for $D = 10$, IIB supergravity

In this section we write a covariant and supersymmetric component level action for IIB supergravity in its canonical formulation, i.e. when the bosonic degrees of freedom are described by A_0, A_2 and A_4, incorporating the dynamics of A_4 according to the method presented in the preceding section.

The component results are obtained from the superspace results of section 2 in a standard fashion setting $\theta = 0 = d\theta$. Whenever we use the same symbols as in section 2 we mean those objects evaluated at $\theta = 0 = d\theta$. In particular the differential d becomes the ordinary differential. Every form can now be decomposed along the vielbeins $e^a = dx^m e_m{}^a$ and the gravitino reduces to $\psi^\alpha = dx^m \psi_m{}^\alpha \equiv e^a \psi_a{}^\alpha$. The supercovariant connection one–form $\omega_a{}^b = dx^m \omega_{ma}{}^b$ is naturally introduced, via equation (2.38) now evaluated at $\theta = 0 = d\theta$, as

$$de^a + e^b \omega_b{}^a = i\bar{\psi}\Gamma^a\psi\,. \tag{4.1}$$

This determines ω as the metric connection, augmented by the standard gravitino bilinears. The supercovariant curvature two–form is now $R_a{}^b = d\omega_a{}^b + \omega_a{}^c\omega_c{}^b$ with

ω given in (4.1). It is also convenient to introduce the A_0, A_2, A_4 supercovariant curvatures as $(n = 1, 3, 5)$

$$F_n = R_n + C_n \,, \tag{4.2}$$

where the R_n are given in (2.7)-(2.9) and (2.12)-(2.14) and the C_n are defined in (2.31)-(2.33). More precisely

$$\begin{aligned} F_1 &= \overline{U} dV - V d\overline{U} + C_1 \,, & (4.3)\\ F_3 &= \overline{U} dA_2 + V d\bar{A}_2 + C_3 \,, & (4.4)\\ F_5 &= dA_4 + i\left(A_2 d\bar{A}_2 - \bar{A}_2 dA_2\right) + C_5 \,. & (4.5) \end{aligned}$$

Since we write the Lagrangian as a ten–form it is also convenient to define the $(10-p)$–forms

$$E^{a_1 \ldots a_p} \equiv \frac{1}{(10-p)!} \epsilon^{a_1 \ldots a_p}{}_{b_1 \ldots b_{10-p}} e^{b_1} \ldots e^{b_{10-p}}. \tag{4.6}$$

In particular $E = \sqrt{-g}\, d^{10}x$.

The action for type IIB supergravity with the canonical fields can now be written as follows:

$$\begin{aligned} S = \int & E_{ab} R^{ab} + \frac{1}{3} E_{abc} \left(i\bar{\psi}\Gamma^{abc} D\psi + \text{c.c.}\right) + 4E_a \left(i\bar{\Lambda}\Gamma^a D\Lambda + \text{c.c.}\right) + \\ & + \frac{1}{4}(F_5 * F_5 + f_4 * f_4) + \frac{1}{2} F_5 dA_4 - \frac{i}{2}\left(A_2 d\bar{A}_2 - \bar{A}_2 dA_2\right) C_5 + \frac{1}{2} C_5^{\Lambda} C_5^{\psi} + \\ & + 2\left[\bar{F}_3 * F_3 + \left(C_7 \bar{F}_3 - \frac{1}{2} C_7 \bar{C}_3 + c.c.\right)\right] + \left(\frac{1}{2}\bar{C}_7{}^{\psi} C_3^{\psi} - 2\bar{C}_7^{\Lambda} C_3^{\psi} + c.c.\right) + \\ & + 2\left[\bar{F}_1 * F_1 + \left(C_9 \bar{F}_1 - \frac{1}{2} C_9 \bar{C}_1 + c.c.\right)\right] - 3E(\bar{\Lambda}\Gamma^a\Lambda)(\bar{\Lambda}\Gamma_a\Lambda) \,. \end{aligned} \tag{4.7}$$

In the first line we have the kinetic terms for the metric, the gravitino and the field Λ. The second line contains the action S_0 of the preceding section, augmented by a term proportional to C_5 which compensates the gauge transformation of $F_5 dA_4$, but is A_4–independent, as required. Since all the other fields are required to be invariant under the transformations I), II) and A_4 appears in (4.7) only in the combination S_0, this action gives as (gauge fixed) equation of motion for A_4 just (3.1), i.e. (2.42).

The third and fourth lines in (4.7) contain, between square brackets, the kinetic and interaction terms for A_2 and A_0 respectively. These particular combinations are just the ones which respect the dualities $A_0 \leftrightarrow A_8$, $A_2 \leftrightarrow A_6$ as we will see in the next section. Variation of (4.7) with respect to A_2 and A_0 produces, as equations of motion, just the Bianchi identities (2.21)-(2.23) of section 2. The remaining terms in the action above are quartic in the fermions and are fixed by supersymmetry, which also fixes the relative coefficients of all the other terms.

The supersymmetry transformations of the fields can again be read from the superspace results. Introducing the transformation parameter $\varepsilon^A = (\varepsilon^\alpha, \overline{\varepsilon^\alpha}, 0)$, the on–shell

SUSY transformations of the component fields are given by covariantized superspace Lie–derivatives of the corresponding superfields, evaluated at $\theta = 0 = d\theta$:

$$\delta_\varepsilon \phi = \left[(i_\varepsilon D + D i_\varepsilon)\, \phi \right]_{\theta=0=d\theta} . \tag{4.8}$$

For the graviton, gravitino, Λ_α and (U, V), we get from (2.38),(2.39),(2.40),(2.25) and the parametrization of R_1

$$\begin{aligned}
\delta_\varepsilon e^a &= i \left(\bar\psi \Gamma^a \varepsilon - \bar\varepsilon \Gamma^a \psi \right) , && (4.9)\\
\delta_\varepsilon \psi &= D\varepsilon + i_\varepsilon D\psi , && (4.10)\\
\delta_\varepsilon \Lambda &= \frac{i}{2} F^a \Gamma_a \bar\varepsilon + \frac{i}{24} F^{abc} \Gamma_{abc} \varepsilon, && (4.11)\\
\delta_\varepsilon U &= -2V \bar\varepsilon \bar\Lambda , && (4.12)\\
\delta_\varepsilon V &= -2U \varepsilon \Lambda. && (4.13)
\end{aligned}$$

The term $i_\varepsilon D\psi$ can be easily evaluated by substituting, in the r.h.s. of (2.39), ψ and $\bar\psi$ respectively with ε and $\bar\varepsilon$.

For what concerns the p–forms, due to gauge invariance and Lorentz invariance, (4.8) would reduce simply to $\delta_\varepsilon A_p = i_\varepsilon dA_p$. However, the presence of the Chern–Simons forms in (2.7)–(2.11) requires compensating SUSY transformations for the potentials A_p. It is convenient to parametrize generic transformations for these potentials in such a way that the curvatures S_n (and R_n) transform covariantly. For later use we give here a complete list of all the combined transformations:

$$\begin{aligned}
\delta A_2 &= \delta_0 A_2 , && (4.14)\\
\delta A_4 &= \delta_0 A_4 + i \left(\delta_0 A_2 \bar A_2 - \delta_0 \bar A_2 A_2 \right) , && (4.15)\\
\delta A_6 &= \delta_0 A_6 + i A_2 \delta_0 A_4 , && (4.16)\\
\delta A_8 &= \delta_0 A_8 - \bar A_2 \delta_0 \bar A_6 + \frac{i}{2} \bar A_2 \bar A_2 \delta_0 A_4 - \frac{1}{4} \left(\bar A_2 \delta_0 A_2 - A_2 \delta_0 \bar A_2 \right) \bar A_2 \bar A_2 , && (4.17)\\
\delta \hat A_8 &= \delta_0 \hat A_8 + \frac{1}{2} \left[\bar A_2 \delta_0 A_6 - \frac{i}{2} A_2 \bar A_2 \delta_0 A_4 + \frac{1}{4} \left(\bar A_2 \delta_0 A_2 - A_2 \delta_0 \bar A_2 \right) A_2 \bar A_2 - \text{c.c.} \right] && (4.18)
\end{aligned}$$

where $\delta_0 A_n$ parametrize generic transformations.

The corresponding (invariant) transformations for the curvatures are:

$$\begin{aligned}
\delta S_3 &= d\delta_0 A_2 , && (4.19)\\
\delta S_5 &= d\delta_0 A_4 + 2i \left(\bar S_3 \delta_0 A_2 - S_3 \delta_0 \bar A_2 \right) , && (4.20)\\
\delta S_7 &= d\delta_0 A_6 + i S_3 \delta_0 A_4 - i S_5 \delta_0 A_2 , && (4.21)\\
\delta S_9 &= d\delta_0 A_8 - \bar S_3 \delta_0 \bar A_6 + \bar S_7 \delta_0 \bar A_2 , && (4.22)\\
\delta \hat S_9 &= d\delta_0 \hat A_8 - \frac{1}{2} \left[\bar S_3 \delta_0 A_6 - S_7 \delta_0 \bar A_2 - \text{c.c.} \right] . && (4.23)
\end{aligned}$$

The transformations for the R_n are easily obtained from their definitions (2.13)–(2.17).

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For supersymmetry transformations we have to choose, here for $n = 2, 4$,

$$\delta_0 A_n = i_\varepsilon S_n\,. \tag{4.24}$$

For the curvatures, this leads to

$$\delta_\varepsilon S_n = (i_\varepsilon D + D i_\varepsilon) S_n\,, \tag{4.25}$$

and

$$\delta_\varepsilon R_n = (i_\varepsilon D + D i_\varepsilon) R_n\,, \tag{4.26}$$

i.e., again to the covariant Lie derivative. Expressing the S_n in terms of the R_n, whose super–space parametrizations are known, in particular $i_\varepsilon R_n = i_\varepsilon(F_n - C_n) = -i_\varepsilon C_n$, the transformations (4.26) can be easily evaluated.

It remains to choose the SUSY transformation law for the auxiliary field $a(x)$. Since this field, being non propagating, has no supersymmetric partner, the simplest choice turns out to be actually the right one. We choose

$$\delta_\varepsilon a = 0\,. \tag{4.27}$$

This concludes the determination of the on–shell SUSY transformation laws for the fields. Due to the chirality condition (2.42), which is an equation of motion of the on–shell superspace approach, some of these transformation laws could change by terms proportional to $F_5 - *F_5$, or equivalently, to $f_4 = i_v(F_5 - *F_5)$. As we will now see, SUSY invariance of the action, and therefore the closure of the SUSY algebra on the transformations I) and II), requires, indeed, such modifications, but, in the present case, only for the gravitino supersymmetry transformation.

In practice the SUSY variation of the action (4.7), which is written as $S = \int \mathcal{L}_{10}$, can be performed by lifting formally the ten–form $\mathcal{L}_{10}$ to superspace, applying then the operator $i_\varepsilon D$ to $\mathcal{L}_{10}$ and using the superspace parametrizations and Bianchi identities of section 2 to show that $\delta_\varepsilon S = \int i_\varepsilon D \mathcal{L}_{10}$ vanishes. The unique term for which this procedure does not work is $\frac{1}{4}\int f_4 * f_4$, because $a(x)$ cannot be lifted to a superfield; therefore the supersymmetry variation of this term has to be performed "by hand". In particular, one has to vary explicitly the vielbeins $e_m{}^a$ contained in v_a.

We give the explicit expression for the SUSY variation of the terms in S which depend on A_4 and v^a (the second line in (4.7)). This will be sufficient to guess the correct off–shell SUSY transformation law for the gravitino (the term proportional to C_5 is included to get a gauge–invariant expression)

$$\begin{aligned}
&\delta_\varepsilon \int \frac{1}{4}(F_5 * F_5 + f_4 * f_4) + \frac{1}{2} F_5 dA_4 - \frac{i}{2}(A_2 d\bar{A}_2 - \bar{A}_2 dA_2) C_5 = \qquad (4.28)\\
&= \int i_\varepsilon \left[\frac{i}{2} K_5 \frac{1}{4!} e^{a_1} \dots e^{a_4} (\bar{\psi}\Gamma^{a_5}\psi) K_{a_5 \dots a_1} + (C_5 - K_5)(dC_5 + 2i\bar{R}_3 R_3) - \frac{1}{2} C_5 dC_5 \right].
\end{aligned}$$

In this expression K_5, which has only components along the bosonic vielbein e^a , is the (self–dual) five–form given in (3.14), in particular $i_\varepsilon K_5 = 0$. The peculiar feature of (4.28) is that the five–form F_5 and the vector v^a appear only in the peculiar combination

$$K_5 \equiv F_5 + v f_4 \,.$$

This suggests to define the off–shell SUSY transformation for the gravitino by making the replacement

$$F^+_{a_1 \dots a_5} \to K_{a_1 \dots a_5} \tag{4.29}$$

in (2.39). The consistency of this replacement with the closure of the SUSY algebra on the transformations I) and II) is a consequence of the facts that K_5 is self–dual as is F_5^+, and that on–shell $F_5^+ = K_5$.

In conclusion, we choose for the gravitino the transformation law

$$\delta_\varepsilon \psi = D\varepsilon + i_\varepsilon (D\psi)_{F_5^+ \to K_5} \,, \tag{4.30}$$

while the transformation laws for all the other fields remain the ones given above.

The ultimate justification for (4.30) stems from the fact that, with this choice, it can actually be checked, with a tedious and long but straightforward computation, that the action given in (4.7) is indeed invariant under supersymmetry.

Finally, let us notice that exactly the same replacements (4.29), (4.30) led to a supersymmetric action also for pure $N = 1$, $D = 6$ supergravity, which contains a chiral two–form gauge potential [6] .

5. $A_2 \leftrightarrow A_6$ duality symmetric action

In this section we will present an action in which the complex six–form potential A_6 and its dual A_2 appear on the same footing. This action will thus depend on e^a, ψ^α, Λ_α, A_0, A_2, A_4 and A_6. The fields A_2 and A_6 and their curvatures are introduced as in (2.7)–(2.11), (2.12)–(2.17) and (2.18)–(2.23), and the associated supercovariant curvatures in ordinary space are again given by

$$F_n = R_n + C_n \,,$$

now for $n = 0, 2, 4, 6$. Since R_3 and R_7 satisfy now their Bianchi identities identically, the dynamics is introduced via the duality relation (2.43)

$$*F_7 = F_3 \,, \tag{5.1}$$

which amounts now to the equation of motion for the system (A_2, A_6); this eventually allows to eliminate A_6 in favour of A_2. In summary, the action we search for has to give rise to the equation (5.1).

We proceed using the tools introduced in section 3; we introduce again the scalar field $a(x)$ and the vector v_a and define the projected forms

$$\begin{aligned} g_2 &\equiv i_v(F_3 - *F_7)\,, \\ g_6 &\equiv i_v(F_7 - *F_3)\,. \end{aligned} \tag{5.2}$$

These complex forms are $SU(1,1)$ singlets and carry $U(1)$ charge $+2$. Due to (3.12) the duality condition (5.1) decomposes then as

$$F_3 - *F_7 = -vg_2 + (*vg_6)\,. \tag{5.3}$$

The projections analogous to f_4, instead, are the complex forms

$$\begin{aligned} f_2 &\equiv Ug_2 - V\bar{g}_2\,, \\ f_6 &\equiv Ug_6 - V\bar{g}_6\,, \end{aligned} \tag{5.4}$$

which are $U(1)$ singlets, and $(f_2, \bar{f}_2)$ and $(if_6, \overline{if_6})$ are $SU(1,1)$ doublets. The duality equation of motion (5.1) is then equivalent to

$$f_2 = 0 = f_6 \qquad \Leftrightarrow \qquad g_2 = 0 = g_6. \tag{5.5}$$

It is also convenient to define

$$g_4 \equiv f_4\,, \tag{5.6}$$

and,

$$K_3 \equiv F_3 + vg_2\,, \tag{5.7}$$

which generalizes the analogous formula (3.14) for A_4,

$$K_5 \equiv F_5 + vg_4\,. \tag{5.8}$$

The action, which involves now (apart from the fermions, the metric and the auxiliary field $a(x)$) the forms A_0, A_2, A_4, A_6, can be written simply as

$$S_{(2,6)} = S + 2\int \bar{g}_2 * g_2\,, \tag{5.9}$$

where S is the basic action given in the previous section. The piece we added is invariant under global $SU(1,1)$ and local $U(1)$, as is S.

We want now to show that $S_{(2,6)}$ exhibits the following features:

i) the fields (A_2, A_6) play a symmetric role;

ii) the dynamics associated to $S_{(2,6)}$ is equivalent to the dynamics described by the original action S; this will be shown through an analysis of the symmetries possessed by $S_{(2,6)}$;

iii) $S_{(2,6)}$ is supersymmetric; to show this we have to find appropriate supersymmetry transformation laws for the fields.

The duality symmetry under $A_2 \leftrightarrow A_6$ is established by extracting from $S_{(2,6)}$ the relevant contributions depending on A_2 and A_6 and by rewriting them in a duality symmetric way. These contributions are the square bracket in the third line of (4.7) and the added term $2\int \bar{g}_2 * g_2$. One finds indeed

$$\begin{aligned} 2 &\left[\bar{F}_3 * F_3 + \left(C_7\bar{F}_3 - \frac{1}{2}C_7\bar{C}_3 + c.c.\right) + \bar{g}_2 * g_2\right] = \\ &= \left[R_3\bar{R}_7 + \bar{C}_7 R_3 + \bar{C}_3 R_7 - v\left(\bar{g}_6 F_3 + \bar{g}_2 F_7\right)\right] + c.c. \end{aligned} \tag{5.10}$$

A *completely* duality symmetric form is forbidden by the appearance of the Chern–Simons forms in the definition of the S_n, and hence of the R_n. In particular, it can be seen that the term $R_3\bar{R}_7 + \bar{R}_3 R_7 = S_3\bar{S}_7 + \bar{S}_3 S_7$, in the absence of Chern–Simons forms, would become a total derivative.

Now we will examine the bosonic symmetries of the action. To this end we consider generic variations of the fields a, A_2 and A_6 and parametrize them as in (4.14). The variation of the action can then be computed to be:

$$\begin{aligned} \delta S_{(2,6)} = \int &-\frac{2v}{\sqrt{-(\partial a)^2}}\left(f_2\bar{f}_6 + \bar{f}_2 f_6 + \frac{1}{4}f_4 f_4\right) d\delta a + \\ &+ \left[d(vf_4) - 2iv\left(\bar{f}_2 S_3 - f_2\bar{S}_3\right)\right]\delta_0 A_4 + \\ &+ \left\{2\left[d(vf_6) + iv\left(f_4 S_3 - f_2 S_5\right)\right]\delta_0\bar{A}_2 + c.c.\right\} + \\ &+ \left\{2d(vf_2)\delta_0 A_6 + c.c.\right\}. \end{aligned} \tag{5.11}$$

From this formula it is not difficult to realize that the action is invariant under the following transformations, which generalize the transformations $I)$ and $II)$ of section 3 (n=2,4,6):

$$I)\quad \delta_0 A_n = \Lambda_{n-1} da, \qquad \delta a = 0\,, \tag{5.12}$$

$$II)\quad \delta_0 A_n = -\frac{\phi}{\sqrt{-(\partial a)^2}} f_n, \qquad \delta a = \phi\,. \tag{5.13}$$

From the invariance $II)$ one can conclude that $a(x)$ is again non propagating and its equation of motion,

$$d\left[\frac{v}{\sqrt{-(\partial a)^2}}\left(f_2\bar{f}_6 + \bar{f}_2 f_6 + \frac{1}{4}f_4 f_4\right)\right] = 0\,, \tag{5.14}$$

can be easily seen to be a consequence of the equations of motion for A_2, A_4, A_6. These equations of motion, which can be read from (5.11), are, in fact, given by

$$d(vf_2) = 0\,, \tag{5.15}$$

$$d(vf_4) = 2iv\left(\bar{f}_2 S_3 - f_2\bar{S}_3\right), \tag{5.16}$$

$$d(vf_6) = iv\left(f_2 S_5 - f_4 S_3\right). \tag{5.17}$$

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The invariances I), which hold also for finite transformations, can be used to reduce these equations to $f_2 = f_4 = f_6 = 0$, in the same way as we did in section 3. Starting from (5.15) one can use Λ_1 to set $f_2 = 0$. At this point the right hand side of (5.16) is zero and one can use Λ_3 to set $f_4 = 0$. With $f_2 = 0 = f_4$, the r.h.s. of (5.17) is also zero and finally one can use Λ_5 to make f_6 vanishing. This leads to

$$\begin{aligned} F_3 &= *F_7\,, \\ F_5 &= *F_5\,. \end{aligned} \tag{5.18}$$

The equations of motion for the other fields, with the these gauge fixings, are actually the same as the ones determined from S, the basic action. This is due to the fact that the added term is *quadratic* in g_2, which vanishes because $f_2 = 0$.

The last issue concerns supersymmetry. We keep for $a(x)$, e^a, A_0, A_2 and A_4 the same SUSY–transformation laws as in the preceding section. In particular, $a(x)$ remains invariant and

$$\delta_0 A_n = i_\varepsilon S_n\,,$$

which holds now for $n = 2, 4, 6$ and fixes also the SUSY–transformation law for A_6, the new field. To find the transformation laws for the fermions it is convenient to proceed as follows. We extract from (4.7) all terms which depend on A_2 and A_4, let us call them[4] all together $\tilde{S}$. Then we perform the SUSY variation of

$$\tilde{S} + 2\int \bar{g}_2 * g_2\,, \tag{5.19}$$

leaving the transformations of the fermions, ψ^α and Λ_α, generic. An explicit computation of this variation leads to the remarkable result that

$$\delta_\varepsilon\left(\tilde{S} + 2\int \bar{g}_2 * g_2\right) = \left(\delta_\varepsilon \tilde{S}\right)_{F_3\to K_3}\,, \tag{5.20}$$

i.e. the variation of (5.19) coincides with the variation of $\tilde{S}$ if we replace in $\delta_\varepsilon \tilde{S}$

$$F_{a_1 a_2 a_3} \to K_{a_1 a_2 a_3}\,, \tag{5.21}$$

where the form K_3 has been defined in (5.7). The main ingredient in this computation is the identity

$$K_3 \equiv F_3 + v g_2 = *\left(F_7 + v g_6\right), \tag{5.22}$$

which is nothing else than (5.3). The result (5.20) and the fact that S is invariant under supersymmetry imply that the action $S_{(2,6)}$ is invariant under supersymmetry if we choose for the fermions the transformation laws:

$$\delta_\varepsilon \psi = D\varepsilon + \left[i_\varepsilon(D\psi)\right]_{\substack{F_5^+\to K_5 \\ F_3\to K_3}}\,, \tag{5.23}$$

$$\delta_\varepsilon \Lambda = \left[i_\varepsilon(D\Lambda)\right]_{F_3\to K_3} = \frac{i}{2} F^a \Gamma_a \bar{\varepsilon} + \frac{i}{24}\left(K^{abc}\Gamma_{abc}\right)\varepsilon\,. \tag{5.24}$$

[4]These are just the ones in the second and third line of (4.7) without the quartic terms in the fermions.

This concludes the proof that the action $S_{(2,6)}$ provides a consistent, manifestly $A_2 \leftrightarrow A_6$ duality invariant Lagrangian formulation for $N = IIB$, $D = 10$ supergravity. It is also manifestly invariant under the local Lorentz group $SO(1,9)$, under local $U(1)$ and under global $SU(1,1)$.

In the next section we extend this procedure to the dualization of the scalars $A_0 = (U, -\overline{V})$.

6. $A_0 \leftrightarrow A_8$ duality symmetric action

The dualization of the scalars follows the strategy developed in the preceding section. We will write an action in which the scalars and the eight–forms appear simultaneously; since the definition of the curvatures of the eight–forms (equations (2.10)–(2.11)) requires the presence of Chern–Simons forms containing $S_7 = dA_6 + \ldots$, the action $S_{(0,8)}$ we search for has to describe also the dynamics of A_6. Thus our starting point will be the action $S_{(2,6)}$ and $S_{(0,8)}$ will involve the whole tower of potentials and dual potentials A_n $(n = 0, 2, 4, 6, 8)$. The S_n, R_n and F_n are introduced as in section 2, at $\theta = 0 = d\theta$.

We recall that we introduce three eight–forms $(A_8, \bar{A}_8, \hat{A}_8)$, a $SU(1,1)$ triplet, and that the equations of motion, which have to be produced by the action $S_{(0,8)}$, are

$$F_9 = *F_1\,, \tag{6.1}$$
$$\hat{F}_9 = 0\,. \tag{6.2}$$

Equation (6.2) fixes the non propagating purely imaginary eight–form $\hat{A}_8$ while equation (6.1) transforms the Bianchi identities for A_8 in equations of motion for A_0 (and vice-versa). $S_{(0,8)}$ has still to produce the duality relations $F_5 = *F_5$ and $F_3 = *F_7$. We already know how to get equation (6.1) from a Lagrangian formulation, just in the same way as we got in the preceding section $F_3 = *F_7$.

On the other hand, equation (6.2), which is not a duality relation between curvatures, needs a (simple) adaptation of our method: we introduce a new auxiliary purely imaginary one–form, which we call $\hat{F}_1$, which is, however, *not* the differential of a scalar and satisfies no Bianchi identity. Then we add to the action the term $2\int \hat{F}_1 * \hat{F}_1$, which will eventually imply the vanishing of $\hat{F}_1$, and impose then with our method the duality relations

$$\begin{aligned} F_9 &= *F_1\,, \\ \hat{F}_9 &= *\hat{F}_1\,. \end{aligned} \tag{6.3}$$

This will finally lead to $\hat{F}_9 = 0$. We choose for $\hat{F}_1$ a vanishing $U(1)$ charge and take it to be a $SU(1,1)$ singlet.

The g_0 and g_8–projections of our duality relations are given by

$$\begin{aligned} g_0 &= i_v(F_1 - *F_9), \qquad & g_8 &= i_v(F_9 - *F_1)\,, \\ \hat{g}_0 &= i_v(\hat{F}_1 - *\hat{F}_9), \qquad & \hat{g}_8 &= i_v(\hat{F}_9 - *\hat{F}_1)\,, \end{aligned} \tag{6.4}$$

such that, as before

$$
\begin{aligned}
F_1 - *F_9 &= -v g_0 + *(v g_8)\,, \\
\hat{F}_1 - *\hat{F}_9 &= -v \hat{g}_0 + *(v \hat{g}_8)\,.
\end{aligned}
\tag{6.5}
$$

The zero and eight–forms defined in (6.4) are all $SU(1,1)$ singlets; g_0 and g_8 have $U(1)$ charge $+4$ and $\hat{g}_0$ and $\hat{g}_8$ have $U(1)$ charge 0. It is also convenient to combine these forms into forms which have all $U(1)$ charge zero and form $SU(1,1)$ triplets:

$$
\begin{aligned}
f_0 &\equiv \overline{U}^2 \bar{g}_0 - \overline{V}^2 g_0 - 2\overline{UV}\hat{g}_0\,, \\
\hat{f}_0 &\equiv \overline{U} V \bar{g}_0 - U\overline{V} g_0 - (|U|^2 + |V|^2)\hat{g}_0\,,
\end{aligned}
\tag{6.6}
$$

and

$$
\begin{aligned}
f_8 &\equiv \overline{U}^2 \bar{g}_8 - \overline{V}^2 g_8 - 2\overline{UV}\hat{g}_8\,, \\
\hat{f}_8 &\equiv \overline{U} V \bar{g}_8 - U\overline{V} g_8 - (|U|^2 + |V|^2)\hat{g}_8\,.
\end{aligned}
\tag{6.7}
$$

$\hat{f}_0$ and $\hat{f}_8$ are purely imaginary and f_0 and f_8 are complex.

Defining the $SU(1,1)$ Lie–algebra elements

$$
F_0 = \begin{pmatrix} \hat{f}_0 & \bar{f}_0 \\ f_0 & -\hat{f}_0 \end{pmatrix}, \tag{6.8}
$$

$$
G_0 = \begin{pmatrix} -\hat{g}_0 & g_0 \\ \bar{g}_0 & \hat{g}_0 \end{pmatrix}, \tag{6.9}
$$

and

$$
F_8 = \begin{pmatrix} \hat{f}_8 & \bar{f}_8 \\ f_8 & -\hat{f}_8 \end{pmatrix}, \tag{6.10}
$$

$$
G_8 = \begin{pmatrix} -\hat{g}_8 & g_8 \\ \bar{g}_8 & \hat{g}_8 \end{pmatrix}, \tag{6.11}
$$

the definitions (6.6) and (6.7) can also be cast in the form

$$
F_0 = W G_0 W^{-1}\,, \qquad F_8 = W G_8 W^{-1}\,, \tag{6.12}
$$

where W is the scalar field matrix defined in (2.1). This makes the transformation properties of F_0 and F_8 as $SU(1,1)$ triplets (adjoint representation) manifest. The duality relations (6.3) are then equivalent to

$$
F_0 = 0 = F_8 \quad \Leftrightarrow \quad G_0 = 0 = G_8\,. \tag{6.13}
$$

The action which depends now on all the fields (and dual fields) of IIB supergravity and also on the auxiliary fields a and $\hat{F}_1$ is :

$$
S_{(0,8)} = S + 2\int \left(\bar{g}_2 * g_2 + \bar{g}_0 * g_0 + \hat{g}_0 * \hat{g}_0 + \hat{F}_1 * \hat{F}_1 \right). \tag{6.14}
$$

The second and third terms in the brackets are analogous to the term quadratic in g_2, which was also present in $S_{(2,6)}$; the last term will eventually imply the vanishing of $\hat{F}_1$. Since the action is quadratic in $\hat{F}_1$ one could think that its equation of motion could be substituted back in the action, leading to the elimination of the auxiliary field $\hat{F}_1$. This is, actually, not the case. In fact, under a generic variation of $\hat{F}_1$ one has:

$$\delta S_{(0,8)} = 2\delta \int \left(\hat{g}_0 * \hat{g}_0 + \hat{F}_1 * \hat{F}_1\right) = 4\int \left(\hat{F}_1 + v\hat{g}_0\right) * \delta\hat{F}_1 \,, \tag{6.15}$$

and the equation of motion for $\hat{F}_1$ is

$$\hat{F}_1 = -v i_v \left(\hat{F}_1 - *\hat{F}_9\right), \tag{6.16}$$

which is equivalent to

$$i_v * \hat{F}_1 = 0 \Leftrightarrow v\hat{F}_1 = 0 \,, \tag{6.17}$$

$$i_v * \hat{F}_9 = 0 \Leftrightarrow v\hat{F}_9 = 0 \,. \tag{6.18}$$

In deriving (6.17)–(6.18) we used (3.12) and the operator identity

$$i_v * = - * v \,, \tag{6.19}$$

which holds on any p–form. Therefore, the component of $\hat{F}_1$ parallel to v remains undetermined and, moreover, one has the constraint (6.18) on $\hat{F}_9$. For this reason the auxiliary field $\hat{F}_1$ can not be eliminated from the action.

When one takes the equations of motion for the eight–forms into account and fixes the symmetries I) and II) (to be discussed now) then, as we will see below, one arrives actually to the equations (6.13) (i.e. (6.3)), and then (6.16) implies indeed $\hat{F}_1 = 0$ and $\hat{F}_9 = 0$, which is precisely what we want.

Now we discuss the bosonic symmetries exhibited by the action $S_{(0,8)}$. In this case, for the transformations of the type I) it is convenient to proceed in a slightly different manner from the preceding section. We consider generic variations δA_n ($n = 0, \ldots, 8, \hat{8}$), including now also the scalars, and take for $\hat{F}_1$ the transformation law

$$\delta\hat{F}_1 = -2i\delta Q \,, \tag{6.20}$$

where Q is the $U(1)$ connection (the fermions, the metric and the field $a(x)$ are not varied). Then the variation of the action can be expressed in terms of the variations of the curvatures as follows:

$$\delta S_{(0,8)} = -\int v \left[2\sum_{n=0,2,6,8}(\bar{g}_n\delta R_{9-n} + c.c.) + g_4\delta R_5 + 4\hat{g}_0\delta\hat{R}_9 - 8i\hat{g}_8\delta Q\right]. \tag{6.21}$$

This vanishes for the transformations ($n = 2, 4, 6, 8, \hat{8}$)

$$Ia) \qquad \delta_0 A_n = \Lambda_{n-1} da, \qquad \delta U = 0 = \delta V \,. \tag{6.22}$$

The reason is that, since $\delta U = 0 = \delta V$, δR_n is given by a linear combination of the δS_n, and from (4.19)–(4.23) one sees that under $Ia)$ all δS_n are proportional to $v = \frac{1}{\sqrt{-(\partial a)^2}} da$. Since in (6.21) all δR_n are multiplied by v they drop out. On the other hand, for $\delta U = 0 = \delta V$ we have $\delta Q = 0$.

It remains to find the symmetries of the type $I)$ for the scalars. The formula for $\delta_0 A_n$ in (6.22) cannot be applied directly to the scalars, but we can observe that, due to gauge invariance, for example for $n = 2$, the transformation (6.22) is equivalent to $\delta A_2 = d\Lambda_1 a$, or

$$\delta A_2 = \Omega_2 a, \quad d\Omega_2 = 0\,.$$

This suggests that the transformations for the scalars analogous to (6.22), should shift them by $a(x)$ multiplied by a constant, or more generally, by some functions of $a(x)$. To be more precise, since the fields U and V are restricted by the condition $|U|^2 - |V|^2 = 1$, we can parametrize generic transformations of these fields by local infinitesimal $SU(1,1)$ transformations

$$\begin{aligned} \delta U &= \gamma \overline{V} + \beta U\,, \\ \delta V &= \gamma \overline{U} + \beta V\,, \end{aligned} \tag{6.23}$$

where γ is complex and β purely imaginary; the matrix

$$M \equiv \begin{pmatrix} \beta & \gamma \\ \bar\gamma & -\beta \end{pmatrix}$$

is indeed an element of the Lie–algebra of $SU(1,1)$. Now, the action $S_{(0,8)}$ is manifestly invariant under global $SU(1,1)$ transformations of the scalars *and* the forms A_n, i.e. when β and γ are constants and one accompanies (6.23) with the corresponding global $SU(1,1)$ transformations of A_2, A_6 and $(A_8, \bar{A}_8, \hat{A}_8)$. This suggests to define the following transformations of the type $I)$ for the scalars (which comprise now also compensating transformations for the forms A_n) as

$$Ib) \qquad \begin{aligned} \delta U &= \gamma \overline{V} + \beta U\,, \\ \delta V &= \gamma \overline{U} + \beta V\,, \\ \delta A_2 &= -\gamma \bar{A}_2 + \beta A_2\,, \\ \delta A_4 &= 0\,, \\ \delta A_6 &= \gamma \bar{A}_6 + \beta A_6\,, \\ \delta A_8 &= 2\left(\gamma \hat{A}_8 - \beta A_8\right), \\ \delta \hat{A}_8 &= \gamma A_8 - \bar\gamma \bar{A}_8\,, \end{aligned}$$

where γ and β are now arbitrary functions of a, i.e. $\beta = \beta(a)$, $\gamma = \gamma(a)$ (In this case we do not use the parametrizations (4.14)–(4.18)). The transformations for A_2 and A_6 and $(A_8, \bar{A}_8, \hat{A}_8)$ are just their transformations as $SU(1,1)$ doublets and triplet respectively, with a–dependent transformation parameters; the transformations $Ib)$ constitute therefore a "quasi–local" $SU(1,1)$ transformation for all the fields where the transformation matrix M depends on x only through the function $a(x)$: $M = M(a)$. It is understood that for $\hat{F}_1$ we choose the transformation (6.20).

From (6.21) it can now be seen that $S_{(0,8)}$ is invariant under $Ib)$. Since the R_n and Q are all invariant under global $SU(1,1)$ transformations, under the quasi–local transformations $Ib)$ we have that δR_n and δQ are all proportional to da, i.e. to v. For example

$$\delta Q = \frac{1}{2i}\left[\overline{UV}\gamma' - UV\bar{\gamma}' + \left(|U|^2+|V|^2\right)\beta'\right] da\,, \tag{6.24}$$

$$\delta R_3 = \left[\left(\overline{U}\beta' - V\bar{\gamma}'\right)A_2 - \left(V\beta' + \overline{U}\gamma'\right)\bar{A}_2\right] da\,, \tag{6.25}$$

where $\beta' = \frac{d\beta}{da}, \gamma' = \frac{d\gamma}{da}$. Therefore $\delta S_{(0,8)}$ vanishes because of the presence of another factor v in (6.21).

Due to the fact that the a–dependent $SU(1,1)$ transformations form a group we can conclude that $S_{(0,8)}$ is invariant also under any *finite* a–dependent $SU(1,1)$ transformation.

For what concerns the transformation $II)$, which should allow to eliminate the auxiliary field $a(x)$, we proceed as in the preceding section. The general variation of $S_{(0,8)}$, parametrized by the variations $\delta_0 A_n$ according to (4.14), by the transformations of the scalars (6.23) and by a generic variation of the field a is:

$$\begin{aligned}
\delta S_{(0,8)} = \int & -\frac{2v}{\sqrt{-(\partial a)^2}}\left[f_0\bar{f}_8 + \bar{f}_0 f_8 + 2\hat{f}_0\hat{f}_8 + f_2\bar{f}_6 + \bar{f}_2 f_6 + \frac{1}{4}f_4 f_4\right] d\delta a + \\
&+ \left[d(vf_4) - 2iv(\bar{f}_2 S_3 - f_2\bar{S}_3)\right]\delta_0 A_4 + \\
&+ \left\{2\left[d(vf_6) + v\left(if_4 S_3 - if_2 S_5 - \bar{f}_0\bar{S}_7 - \hat{f}_0 S_7\right)\right]\delta_0\bar{A}_2 + c.c.\right\} + \\
&+ \left\{2\left[d(vf_2) + v\left(\bar{f}_0\bar{S}_3 - \hat{f}_0 S_3\right)\right]\delta_0\bar{A}_6 + c.c.\right\} + \\
&+ 2\left[d(vf_8) + v\left(2\hat{f}_0 S_9 - 2f_0\hat{S}_9 - \bar{f}_6\bar{S}_3 + \bar{f}_2\bar{S}_7 + c.c.\right)\right]\gamma + \\
&+ 2\left[2d(v\hat{f}_8) + v\left(2\bar{S}_9 f_0 - \bar{S}_3 f_6 - \bar{S}_7 f_2 - c.c.\right)\right]\beta \\
&+ \left[2d(v\bar{f}_0)\delta A_8 + c.c.\right] + 4d(v\hat{f}_0)\delta\hat{A}_8\,.
\end{aligned} \tag{6.26}$$

From this formula, which generalizes (5.11), one sees that $S_{(0,8)}$ is indeed invariant under the transformations of the type $II)$ given by

$$II) \qquad \begin{aligned}
\delta a &= \phi\,, \\
\delta_0 A_n &= -\frac{\phi}{\sqrt{-(\partial a)^2}} f_n, \qquad (\text{n} = 2,4,6,8,\hat{8})\,, \\
\delta U &= -\frac{\phi}{\sqrt{-(\partial a)^2}}(U\hat{f}_0 + \overline{V}\bar{f}_0)\,, \\
\delta V &= -\frac{\phi}{\sqrt{-(\partial a)^2}}(V\hat{f}_0 + \overline{U}\bar{f}_0)\,,
\end{aligned} \tag{6.27}$$

where for $\hat{F}_1$ we choose again (6.20) as transformation law. These transformations allow again to eliminate the auxiliary field $a(x)$.

The transformations I), instead, allow to reduce the equations of motion for the bosons to the first order duality equations (2.42)–(2.45). From the general variation of $S_{(0,8)}$ (6.26) one can, in fact, read the equations of motion for $A_8, \hat{A}_8, A_6, A_4, A_2$ and the scalars which are respectively given by:

$$d(vf_0) = 0 = d(v\hat{f}_0)\,, \tag{6.28}$$
$$d(vf_2) = v\left(\hat{f}_0 S_3 - \bar{f}_0 \bar{S}_3\right), \tag{6.29}$$
$$d(vf_4) = 2iv\left(\bar{f}_2 S_3 - f_2 \bar{S}_3\right), \tag{6.30}$$
$$d(vf_6) = v\left(\hat{f}_0 S_7 + \bar{f}_0 \bar{S}_7 + if_2 S_5 - if_4 S_3\right), \tag{6.31}$$
$$d(vf_8) = v\left(2f_0 \hat{S}_9 - 2\hat{f}_0 S_9 + \bar{f}_6 \bar{S}_3 - \bar{f}_2 \bar{S}_7\right), \tag{6.32}$$
$$d(v\hat{f}_8) = v\left(f_2 \bar{S}_7 + f_6 \bar{S}_3 - 2f_0 \bar{S}_9 - c.c.\right). \tag{6.33}$$

To reduce these equations to the self–duality equations, one has to start from (6.28) and use the Ib) invariances in their finite form, i.e. the a–dependent $SU(1,1)$ transformations. In more detail, (6.28), which can also be written as

$$d(vF_0) = 0\,, \tag{6.34}$$

where F_0 is the $SU(1,1)$ Lie–algebra valued matrix given in (6.8), has the general solution

$$F_0 = \sqrt{-(\partial a)^2} \cdot \Sigma(a)\,, \tag{6.35}$$

where $\Sigma(a)$ is an arbitrary Lie–algebra valued matrix which depends on x^m only through $a(x)$. We want now to show that the r.h.s. of (6.35) can be cancelled by a finite a–dependent $SU(1,1)$ transformation $\Lambda(a)$ (in the doublet representation of $SU(1,1)$). To see in which way F_0 transforms we remember that $F_0 = WG_0W^{-1}$ and observe that, due to its definition (6.9) and to (2.2)

$$G_0 = i_v\left[W^{-1}dW - (\hat{F}_1 + 2iQ)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - *\begin{pmatrix} -\hat{F}_9 & F_9 \\ \bar{F}_9 & \hat{F}_9 \end{pmatrix}\right]. \tag{6.36}$$

Thanks to (6.20) the second term in this expression is invariant as is also the last one because, under $\Lambda(a)$, $(F_9, \bar{F}_9, \hat{F}_9)$ transform by terms proportional to v, and one has the operator identity $i_v * = - * v$.

Under a $SU(1,1)$ transformation we have

$$W \to \Lambda(a)W,$$

and therefore, from the first term in (6.36)

$$G_0 \to G_0 - \sqrt{-(\partial a)^2}W^{-1}\left(\Lambda^{-1}\frac{\delta\Lambda}{\delta a}\right)W\,.$$

This leads for F_0 to the transformation

$$F_0 \to \Lambda F_0 \Lambda^{-1} - \sqrt{-(\partial a)^2}\left(\frac{\delta\Lambda}{\delta a}\Lambda^{-1}\right)$$

and (6.35) transforms into

$$\Lambda F_0 \Lambda^{-1} = \sqrt{-(\partial a)^2}\left[\Sigma(a) + \frac{\delta\Lambda(a)}{\delta a}\Lambda^{-1}(a)\right]. \tag{6.37}$$

Since $\frac{\delta\Lambda}{\delta a}\Lambda^{-1}$ belongs to the Lie–algebra of $SU(1,1)$, as does $\Sigma(a)$, the equation

$$\frac{\delta\Lambda(a)}{\delta a}\Lambda^{-1}(a) = -\Sigma(a) \tag{6.38}$$

fixes, indeed, consistently and uniquely $\Lambda(a)$, modulo a global $SU(1,1)$ transformation. Choosing, therefore, $\Lambda(a)$ as in (6.38) we get $F_0 = 0$, i.e.

$$f_0 = 0 = \hat{f}_0\,. \tag{6.39}$$

Since the transformation $\Lambda(a)$ leaves the equations of motion (6.29)–(6.33) invariant (or sends them into linear combinations), (6.29) becomes now, due to (6.39)

$$d(vf_2) = 0.$$

This has the solution:

$$vf_2 = dad\tilde{\Lambda}_1\,, \tag{6.40}$$

and the transformations Ia), with only Λ_1 non–vanishing and $\Lambda_1 = \tilde{\Lambda}_1$ reduce (6.40) to $f_2 = 0$. This transformation leaves (6.30)–(6.33) invariant and maintains also (6.39). Now one can proceed with (6.30), whose r.h.s. is now vanishing, to set also f_4 to zero and so on. In this way one can use the transformations Ia) and the equations of motion (6.28)–(6.33) to have finally for *all the values of* n $f_n = 0$, or, equivalently

$$g_n = 0. \tag{6.41}$$

But this is equivalent to the duality relations

$$\begin{aligned}
n &= 4 \quad & F_5 &= *F_5\,,\\
n &= 2,6 \quad & F_7 &= *F_3\,,\\
n &= 0,8 \quad & F_9 &= *F_1\,,\\
n &= \hat{0},\hat{8} \quad & \hat{F}_9 &= *\hat{F}_1\,.
\end{aligned}$$

The equation of motion for $\hat{F}_1$ given in (6.16), i.e. $\hat{F}_1 = -v\hat{g}_0$, leads then finally to

$$\hat{F}_9 = 0 = \hat{F}_1.$$

This concludes the proof that the dynamics described by the action $S_{(0,8)}$ is equivalent to the dynamics of $D = 10$, IIB supergravity.

The last issue is supersymmetry. The supersymmetry variation of $S_{(0,8)}$ can again be performed in a standard way, following the procedure of section 5. The SUSY–variation of e^a is the standard one and again we choose $\delta_\varepsilon a = 0$. The variation of the forms A_n $(n = 2, 4, 6, 8, \hat{8})$ is parametrized by (4.14) with

$$\delta_0 A_n = i_\varepsilon S_n,$$

as in the preceding section. The supersymmetry transformations of the fermions are obtained from the standard ones, given in section 4, through the replacements:

$$F_3 \to K_3 \equiv F_3 + v g_2\,,$$
$$F_1 \to K_1 \equiv F_1 + v g_0\,.$$

More precisely:

$$\delta\psi = D\varepsilon + (i_\varepsilon D\psi)_{\substack{F_5^+ \to K_5 \\ F_3 \to K_3}}\,, \tag{6.42}$$

$$\delta\Lambda = (i_\varepsilon D\Lambda)_{\substack{F_3 \to K_3 \\ F_1 \to K_1}} = \frac{i}{2}(K^a\Gamma_a)\bar{\varepsilon} + \frac{i}{24}\left(K^{abc}\Gamma_{abc}\right)\varepsilon\,. \tag{6.43}$$

Notice that the modification of the supersymmetry transformation laws are proportional to the g_n, and vanish therefore on–shell, and that $\hat{g}_0$ does not enter in these modifications since its partner $\hat{F}_1$ is not present in the standard formulation of IIB supergravity.

It remains to fix the SUSY transformation of $\hat{F}_1$. To this order we remember that under a generic variation of $\hat{F}_1$ we have, from (6.15)

$$\delta S_{(0,8)} = 4\int \hat{K}_1 * \delta\hat{F}_1\,, \tag{6.44}$$

where

$$\hat{K}_1 \equiv \hat{F}_1 + v\hat{g}_0\,, \tag{6.45}$$
$$\delta\hat{F}_1 = dx^m \delta\hat{F}_m\,. \tag{6.46}$$

This means that $\delta_\varepsilon\hat{F}_1$ can be used to cancel from $\delta_\varepsilon S_{(0,8)}$ all "residual" terms which are proportional to $\hat{K}_1$, i.e. terms which are proportional to the equations of motion of $\hat{F}_1$. Such residual terms are, actually, present and it turns out that one has to choose

$$\delta_\varepsilon\hat{F}_1 = \left[-W_{(ab)} + \frac{1}{2}\eta_{ab}W_c{}^c\right] e^a\hat{K}^b - 2v\left[\bar{g}_0(\varepsilon\Lambda) - g_0(\bar{\varepsilon}\bar{\Lambda})\right]\,, \tag{6.47}$$

where $W_{ab} = i\left[\bar{\psi}_a\Gamma_b\varepsilon - \bar{\varepsilon}\Gamma_b\psi_a\right]$. With this SUSY transformation for $\hat{F}_1$ and the transformations for the other fields defined above one can check that $S_{(0,8)}$ is indeed supersymmetric:

$$\delta_\varepsilon S_{(0,8)} = 0\,.$$

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The on–shell consistency of (6.47) can be verified as follows. The equation of motion for $\hat{F}_1$ is

$$\hat{K}_1 \equiv \hat{F}_1 + v\hat{g}_0 = 0$$

and (6.47) reduces to:

$$\delta\hat{F}_1 = -2\frac{da}{\sqrt{-(\partial a)^2}}\left(\bar{g}_0(\varepsilon\Lambda) - g_0(\bar{\varepsilon}\bar{\Lambda})\right). \tag{6.48}$$

On the other hand $\hat{K}_1 = 0$ gives

$$\hat{F}_1 = -\frac{da}{\sqrt{-(\partial a)^2}}\hat{g}_0\,. \tag{6.49}$$

Since on–shell we have

$$G_0 = \begin{pmatrix} -\hat{g}_0 & g_0 \\ \bar{g}_0 & \hat{g}_0 \end{pmatrix} = \sqrt{-(\partial a)^2}\cdot W^{-1}\Sigma(a)W\,,$$

the factor $\sqrt{-(\partial a)^2}$ disappears in (6.48) and (6.49). Since, moreover, $\delta_\varepsilon a = 0$, in computing the variation of the r.h.s. of (6.49) one has only to vary the fields U and V contained in W, and the result can easily be seen to coincide with (6.48).

7. Concluding remarks

In this paper we presented an action which produces correctly the dynamics of $D = 10, N = IIB$ supergravity, (4.7), and possesses all relevant symmetries. A non manifestly Lorentz invariant action can be obtained by setting, for example, $a(x) = n_m x^m$; this eliminates the unique auxiliary field but leaves the action still invariant under the transformations of the type I). These transformations allow to recover the self–duality condition for the chiral four–form. Apart from fundamental reasons, the knowledge of this action could for example be useful when one couples the relevant Dp–branes to IIB supergravity, i.e. considers an action of the form $S^{Sugra}_{IIB} + S^{\sigma-model}_{Dp}$.

We presented also actions which are manifestly invariant under the interchange of the basic forms and their Hodge–duals. In this case the dual fields can be eliminated upon gauge–fixing the transformations I) while it is not possible to eliminate the basic fields in favour of the dual ones. With the gauge fixings we performed in the paper, one is essentially forced to interpret the Bianchi identities of the dual fields as equations of motion of the basic fields. As we will sketch now, however, in a perturbative treatment one can actually treat the basic fields and their duals in a symmetric fashion. We will outline the procedure in the case of $A_2 \leftrightarrow A_6$, at the linearized level, i.e. for vanishing fermions, for a flat metric and choosing for simplicity $U = 1$ and $V = 0$.

First of all we choose a gauge fixing for the transformations II) according to $a = n_m x^m$. This implies

$$v_m = n_m\,, \qquad dv = 0\,. \tag{7.1}$$

The equations of motion for A_2 and A_6, eqs. (5.15) and (5.17), reduce then to

$$v\, di_v(dA_2 - *dA_6) = 0\,, \tag{7.2}$$
$$v\, di_v(dA_6 - *dA_2) = 0\,. \tag{7.3}$$

Appropriate gauge fixings for the transformations I) and for the gauge invariances $\delta A_n = d\Lambda_{n-1}$ are

$$d * A_2 = 0 = d * A_6\,, \tag{7.4}$$
$$i_v A_2 = 0 = i_v A_6\,. \tag{7.5}$$

This reduces (7.2) and (7.3) further to

$$T\partial_v A_2 + (\Box + \partial_v^2)A_6 = 0 = T\partial_v A_6 + (\Box + \partial_v^2)A_2\,, \tag{7.6}$$

where $\partial_v = v^m\partial_m$ and $T = *vd$. In particular, the operator T satisfies, on forms which are constrained by (7.4) and (7.5),

$$T^2 = \Box + \partial_v^2\,.$$

Using this one can combine the equations in (7.6) to get

$$\Box A_2 = 0 = \Box A_6\,,$$

which are the correct equations for massless fields, and

$$TA_2 + \partial_v A_6 = 0 = TA_6 + \partial_v A_2\,, \tag{7.7}$$

which is a residual constraint on the polarizations. To see which polarizations survive we go to momentum space, $A_n(x) \to a_n(p)$, and choose $n^m = (1, 0, \cdots, 0)$. Splitting the ten–dimensional index m as $m = (0, r)$, (7.5) says that only space–like indices survive in the polarizations and (7.4) gives the transversality conditions

$$p^r a_{rs} = 0 = p^r a_{rr_1r_2r_3r_4r_5}\,. \tag{7.8}$$

Therefore, both a_2 and a_6 carry 28 degrees of freedom, but the residual conditions (7.7) identify now these polarizations giving in momentum space the single constraint

$$a_{r_1\cdots r_6} = \frac{1}{2}\varepsilon_{r_1\cdots r_6}{}^{s_1s_2s_3}\frac{p_{s_1}}{|\vec{p}|}a_{s_2s_3}\,.$$

This constitutes just a check of the fact that our $A_2 \leftrightarrow A_6$ duality invariant action propagates the correct degrees of freedom.

On the other hand, the gauge fixings (7.4),(7.5) are symmetric under $A_2 \leftrightarrow A_6$ and they could also be used in a functional integral. Clearly, for ten dimensional theories a functional integral is of very little relevance; let us mention, however, that using these gauge–fixings one can indeed perform the (duality symmetric) integration over the two gauge fields in the four–dimensional duality symmetric action for Maxwell's equations coupled to electric and magnetic sources proposed in [17], and the resulting effective interaction for electric and magnetic charges turns out to be the expected one and, in particular, independent on v [9].

Just the same gauge–fixings (7.4),(7.5) appear also appropriate in ten dimensions for what concerns the determination of the Lorentz–anomaly due to A_4, in a perturbative functional integral approach.

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